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LOOSE BLOCK INDEPENDENCE

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ABSTRACT

A finite state stationary process is defined to be loosely block independent if long blocks are almost independent in the $\bar{f}$ sense. We show that loose block independence is preserved under Kakutani equivalence and $\bar{f}$ limits. We show directly that any loosely block independent process is the $\bar{f}$ limit of Bernoulli processes and is a factor of a process which is Kakutani equivalent to a Bernoulli shift. The existing equivalence theory then yields that the loosely block independent processes are exactly the loosely Bernoulli (or finitely fixed) processes.

1. Introduction

In their studies of Kakutani equivalence, Jack Feldman and Anatole Katok independently introduced a monotone matching metric, now called $\bar{f}$. They use it to define loosely Bernoulli, a notion analogous to very weak Bernoulli with the new metric replacing Ornstein's $\bar{d}$. The loosely Bernoulli property is seen to play a fundamental role in the equivalence theory since a process is loosely Bernoulli if and only if it is Kakutani equivalent either to a Bernoulli shift (independent identically distributed process) or to a rotation of the circle [1, 3, 6].

In this article we introduce another $\bar{f}$ concept called loose block independence, analogous to Paul Shield's $\bar{d}$ notion of almost block independence. This property is stable under building towers, inducing on sets, taking factors, and taking $\bar{f}$-limits. The collection of loosely block independent processes is easily seen to contain all Bernoulli shifts and circle rotations and hence must contain their entire Kakutani equivalence classes.

Moreover, the Bernoulli processes are $\bar{f}$-dense in the class of loosely block independent processes; and every loosely block independent process is a factor of a transformation which is Kakutani equivalent to a Bernoulli shift. The standard equivalence theory [6] yields immediately that the loosely block

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independent processes are exactly those which are Kakutani equivalent to Bernoulli shifts or to rotations.

2. Definitions

Let A be a finite index set. Throughout this article T will be an invertible measure preserving transformation on a probability space (X,μ); and $P = \{P_a : a \in A\}$ will be a measurable partition of X. The process (T, P) determines a distribution measure (which we shall also denote by μ) on $A^{\mathbb{Z}}$, the space of all doubly infinite sequences from A. For n a positive integer, the projection of μ onto A^n will be denoted by μ_n.

We recall the following definitions. For $B \subseteq X$, $\mu(B) > 0$, and $x \in B$, the return time is $n(x) = \inf\{k > 0 : T^k x \in B\}$. This function is finite a.e. and defines the induced transformation T_B by $T_B(x) = T^{n(x)}x$. The notion dual to that of an induced transformation is that of a tower transformation. If h is an integrable function on X with values in the set of positive integers, we define the tower space X^h by $X^h = \{(x, i) : 1 \leqq i \leqq h(x)\}$, with normalized measure inherited from that on X. The tower transformation T^h is defined by $T^h(x, i) = (x, i+1)$ for $1 \leqq i < h(x)$ and $T^h(x, h(x)) = (Tx, 1)$. A partition P on X extends to a standard partition P^h on X^h consisting of the complement of the base of the tower adjoined to the collection of sets in P.

Transformations T and S on spaces (X,μ) and (Y,ν), respectively, are said to be Kakutani equivalent if there exist sets $B \subseteq X$, $\mu(B) > 0$ and $C \subseteq Y$, $\nu(C) > 0$ with T_B isomorphic to S_C. Equivalently, in the dual formulation T and S are Kakutani equivalent if there exist tower functions h and k with T^h isomorphic to S^k.

In this paragraph we sketch the essentials of the $\bar{f}$-metric. Let A be a finite index set and let n be a positive integer. For $x, y \in A^n$ we define $\bar{f}(x, y) = 1 - k/n$, where k is the largest integer for which there exist sequences $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$ such that $x(i_l) = y(j_l)$ for $1 \leqq l \leqq k$. For processes (T, P) and (S, Q) indexed by the same set A, with corresponding distributions μ and ν on $A^{\mathbb{Z}}$, we set $\bar{f}_n((T, P), (S, Q)) = \bar{f}(\mu_n, \nu_n) = \inf\{\varepsilon : \text{there is a measure } \rho$ on $A^n \times A^n$ with marginals μ_n and ν_n such that $\rho\{(x, y) : \bar{f}_n(x, y) < \varepsilon\} > 1 - \varepsilon\}$. We further define $\bar{f}((T, P), (S, Q)) = \bar{f}(\mu, \nu) = \limsup \bar{f}_n((T, P), (S, Q))$.

Given a process (T, P), the independent N-blocking, $(T, P)_N$, of (T, P) is the (usually non-stationary) process associated with the product measure $(\mu_N)^{\mathbb{Z}}$.

Definition. A stationary process (T, P) is loosely block independent (LBI) if for every $\varepsilon > 0$ there exists an integer M so that $\bar{f}((T, P), (T, P)_N) < \varepsilon$ for all $N \geqq M$.

3. Kakutani stability and $\bar{f}$-closure

PROPOSITION 1. *The class of* LBI *processes is $\bar{f}$-closed.*

PROOF. We claim that if $\bar{f}((T,P),(S,Q))<\varepsilon$ and if (S,Q) is LBI, then for all N sufficiently large we have $\bar{f}((T,P),(T,P)_N)<3\varepsilon$. For if M is sufficiently large, then for all $N \geqq M$ we have $\bar{f}_N((T,P),(S,Q))<\varepsilon$. Moreover, for the same N we have $\bar{f}((T,P)_N,(S,Q)_N)<\varepsilon$. However, if M is also so large that $\bar{f}((S,Q)_N,(S,Q))<\varepsilon$ for all $N \geqq M$, the triangle inequality yields $\bar{f}_N((T,P),(T,P)_N)<3\varepsilon$.

PROPOSITION 2. *If (T,P) is* LBI *and Q is any partition $Q \subseteq \bigvee_{-\infty}^{\infty} T^iP$, then the factor process (T,Q) is also* LBI.

PROOF. It is clear that for any positive integer k, the process $(T, \bigvee_{-k}^{k} T^iP)$ is LBI. Hence, for any partition $\bar{Q} \subseteq \bigvee_{-k}^{k} T^iP$, we also have that $(T,\bar{Q})$ must be LBI. However, since $Q \subseteq \bigvee_{-\infty}^{\infty} T^iP$, there exists a sequence of partitions $\{\bar{Q}_k\}$ with $\bar{Q}_k \subseteq \bigvee_{-k}^{k} T^iP$ such that $\bar{f}((T,\bar{Q}_k),(T,Q)) \to 0$.

REMARK. A transformation T has one generating partition P for which (T,P) is LBI if and only if (T,Q) is LBI for every generating partition Q. Consequently it makes sense in this case to refer to T as loosely block independent.

The following two propositions show that being LBI is an invariant of Kakutani equivalence.

PROPOSITION 3. *If (T,P) is an* LBI *process, B is a set with $B \in \bigvee_{-\infty}^{\infty} T^iP$, and $Q=\{Q_c : c \in C\} \subseteq \bigvee_{-\infty}^{\infty} T^iP$ is a finite partition of B, then the process (T_B, Q) is* LBI.

PROOF. By the preceding proposition we may assume that the partition P consists of the complement of B adjoined to the set of atoms given by the partition Q on B. Since LBI processes are ergodic, the result follows immediately from the ergodic theorem and the following observation. For arbitrary N, let r_N be the greatest integer in $N\mu(B)$. Then for large N the distribution $(\mu_{r_N})^{Z}$ on C^{Z} given by (T_B,Q) is $\bar{f}$-close to the distribution $(\mu_{r_N})^{Z}$ on C^{Z} obtained from (T,P) by deleting outputs in the complement of B and renormalizing.

PROPOSITION 4. *Let (T,P) be* LBI *and let h be a function measurable with respect to $\bigvee_{-\infty}^{\infty} T^iP$, taking values in the positive integers. If T^h represents the transformation on the tower and P^h is the standard partition of the tower, then the process (T^h,P^h) is* LBI.

PROOF. As before, we may assume that h is measurable with respect to the partition P itself. Reversing the argument in the proof of the preceding proposition yields the desired result.

4. Bernoulli processes are f-dense

It is almost immediate from the definition that Bernoulli shifts and rotations are LBI. Moreover, since the LBI property is stable under Kakutani equivalence and $\bar{f}$-limits, the set of LBI processes must contain the $\bar{f}$-closure of the classes Kakutani equivalent to a rotation or a Bernoulli shift. Indeed, as we show in this section, the LBI processes contain only processes in this closure; the family of LBI processes is the smallest $\bar{f}$-closed class containing the Bernoulli processes.

We recall the following definitions [5]. For a process (T, P) taking values in $\{0,1\}^Z$, an N-cell is an event characterized by the occurrence of a sequence $011\cdots10$ of length $N+1$, consisting of zero, followed by $(N-1)$ ones, followed by zero. An (N,δ)-process is a stationary binary process such that the probability of being in an N-cell at time-zero is at least $(1-\delta)$. As shown in [5], for any choice of N and δ, there exists an (N,δ)-process which is a factor of a Bernoulli shift.

PROPOSITION 5. *For each $\varepsilon>0$ and* LBI *process (T,P), there exists a process (S,Q), which is a factor of a Bernoulli shift, such that $\bar{f}((T,P),(S,Q))<\varepsilon$. That is, the factors of Bernoulli shifts are $\bar{f}$-dense in the* LBI *processes.*

PROOF. Choose N such that $\bar{f}((T,P),(T,P)_N)<\varepsilon/2$. Let (S_1, W_1) be an $(N,\varepsilon/2)$-Bernoulli process. Let (S_0, W_0) be the (infinite entropy) Bernoulli shift determined by the product measure λ^Z, where λ is Lebesgue measure on the unit interval $[0,1]$. Let μ denote the distribution on A^Z determined by the process (T,P). Choose any block code $\hat{h}:[0,1]^N\to A^N$ such that $\lambda_N(\hat{h}^{-1}(a))=\mu_N(a)$ for each $a\in A^N$; and define a shift invariant measure preserving map ϕ, called an almost block code, from the product $(S_0, W_0)\times(S_1,W_1)$ to A^Z as follows: the time-zero output $(\phi(w_0,w_1))_0$ will be $\hat{h}((w_0)_{2-i}^{N+1-i})_i$, provided the time-zero output of w_1 lies in the i-th position of an N-cell, otherwise $(\phi(w_0,w_1))_0=a_1$. The process given by $\phi((S_0,W_0)\times(S_1,W_1))$ is the desired (S,Q), for it is clear that $\bar{f}((S,Q),(T,P)_N)<\varepsilon/2$. The triangle inequality then yields $\bar{f}((S,Q),(T,P))<\varepsilon$.

COROLLARY. *The* LBI *processes are the $\bar{f}$-closure of the Bernoulli processes (or very weak Bernoulli processes, or finitely determined processes, or almost block independent processes).*

LEMMA 6. *Let (S, Q) be a process on sequence space $B^{\mathbf{Z}}$. Let ϕ and ψ be almost block codes from $B^{\mathbf{Z}}$ to $C^{\mathbf{Z}}$ satisfying the following conditions:*

(1) both ϕ and ψ take their block structure from the same (N, δ)-process and agree outside N-cells, and

(2) $\bar{f}_N(\phi(x)_1^N, \psi(x)_1^N) < \varepsilon$ for all sequences x from (S, Q) having the time-zero output located at the beginning of an N-cell, except on a subset of conditional measure ε.

Then there exists a transformation S', which agrees with S except on a set of measure ε, and a partition Q', which agrees with $\phi^{-1}(C)$ except on a set of measure $2(\varepsilon + 1/N)$, such that the process (S', Q') has the same distribution as the process $\psi(S, Q)$. (Note that S' and Q' are measurable with respect to $\bigvee_{-\infty}^{\infty} S^i Q$, but not necessarily with respect to the sub-σ-algebra generated by ϕ.)

PROOF. For fixed x, the strings $\phi(x)_2^N$ and $\psi(x)_2^N$ have a monotone matching yielding their $\bar{f}$-distance. This monotone match then assigns each integer index k between 2 and N to one of two classes: $k = m_\phi(j)$ if $\phi(x)_k$ is the j-th element matched, whereas $k = u_\phi(j)$ if $\phi(x)_k$ is the j-th element unmatched; and similarly for $m_\psi(j)$ and $u_\psi(j)$. Define a permutation $\bar{S}$ of the entries in $\psi(x)_2^N$ in the following manner: set $b_k = \psi(x)_{u_\psi(j)}$ if $k = u_\phi(j)$; alternatively set $b_k = \psi(x)_{m_\psi(j)}$ if $k = m_\phi(j)$. Then

$$|\{k : b_k \neq \phi(x)_k\}|/(N-1) \leqq \bar{f}_{N-1}(\phi(x)_2^N, \psi(x)_2^N).$$

Moreover, there exists a set K of indices between 2 and N such that the induced map given by the permutation $\bar{S}$ above, restricted to K, agrees with the restriction of the permutation taking k to $k+1$, where the cardinality $|K|$ is at least $[1 - \bar{f}_{N-1}(\phi(x)_2^N, \psi(x)_2^N)](N-1)$. The entries in b_1^N determine the partition Q' and the permutation $\bar{S}$ determines the map S'.

THEOREM 7. *If (T, P) is an* LBI *process taking values in $A^{\mathbf{Z}}$, then there exists a process (S, Q) with the same distribution as (T, P), where S is Kakutani equivalent to a factor of an infinite entropy Bernoulli shift.*

PROOF. Fix an arbitrary finite alphabet $\bar{W}$ and set $W_i = \bar{W}$ for $i = 1, 2, \cdots$. Let $W_1 \times W_2 \times \cdots$ be the generating partition of an infinite entropy Bernoulli shift $\bar{B}$. Let $\bar{\bar{B}}$ be an infinite entropy Bernoulli shift which is uniformly distributed on $[0, 1]$. Our desired Bernoulli shift B will be the product of $\bar{B}$ and $\bar{\bar{B}}$; and W will denote a generating partition for it. For each positive integer N, the partition W^N will have no atoms; and this fact will facilitate the construction of block codes of length N. On the other hand, we will use the independent factors of $\bar{B}$ generated by the W_i to construct (N, δ) processes for larger and larger N and smaller and smaller δ.

Let $\varepsilon_i \downarrow 0$ satisfy $\Sigma\, \varepsilon_i < \infty$, and choose N_1 such that $\bar{f}((T, P), (T, P)_M) < \varepsilon_1^2/9$ for $M \geqq N_1$. Take $\xi_1(B, W_1)$ to be an (N_1, ε_1)-process, and let ϕ_1 be the almost block code described in Proposition 5. That is, ϕ_1 takes its block structure from $\xi_1(B, W_1)$ and assigns names inside blocks from a block code so that $\bar{f}((T, P), \phi_1(B, W)) < \varepsilon_1^2/9$. Note that ϕ_1 may be chosen to depend only on finitely many, say k_1, coordinates. (See [5].) Denote by (S_1, Q_1) the process $\phi_1(B, W)$.

Now choose N_2 such that $k_1/N_2 < \varepsilon_2$, and such that for $M \geqq N_2$ we have $\bar{f}((T, P), (T, P)_M) < \varepsilon_2^2/9$. Let $\xi_2(B, W_2)$ be an (N_2, ε_2)-process. Then there are two natural almost block codes, which we denote by ψ_2 and ϕ_2, which take their structure from $\xi_2(B, W_2)$ and satisfy the following conditions:

(1) ψ_2 agrees with ϕ_1 inside N_2-blocks, except in the first or last k_1 places, where it is defined arbitrarily;

(2) ϕ_2 has the same distribution as (T, P) inside N_2-blocks;

(3) since $\bar{f}((T, P), \phi_1(B, W)) < \varepsilon_1^2/9$, the map ϕ_2 can also be made to satisfy $\bar{f}_{N_2}(\psi_2(x)_1^{N_2}, \phi_2(x)_1^{N_2}) < \varepsilon_1$ for all sequences x with time-zero at the beginning of an N_2-cell, except for a subset of conditional measure less than ε_1.

Lemma 6 guarantees that by changing B and $\phi_1^{-1}(A)$ on sets of measure ε_1, we obtain a process (S_2, Q_2) with the same distribution as $\phi_2(B, W)$. The change does not affect the independent distribution of $W_3 \times W_4 \times \cdots$.

We continue inductively. At the n-th stage we have a process $(S_{n-1}, Q_{n-1}) = \phi_{n-1}(B, W)$, where ϕ_{n-1} depends on k_{n-1} coordinates of the output of the product of $\bar{\bar{B}}$ and the first $(n-1)$ copies $W_1 \times \cdots \times W_{n-1}$. Because (T, P) is LBI, there exists an integer N_n such that $k_{n-1}/N_n < \varepsilon_n$, and such that $\bar{f}((T, P), (T, P)_M) < \varepsilon_n^2/9$ for all $M \geqq N_n$. Let $\xi_n(B, W_n)$ be an (N_n, ε_n)-process. (The output sequences of the process (B, W_n) may have been changed in the process of transforming S_1 to S_{n-1}; however, it remains an independent process.) Use $\xi_n(B, W_n)$ to get an N_n-block structure for the almost block codes ψ_n and ϕ_n satisfying the following conditions:

(1) ψ_n agrees with ϕ_{n-1} except for locations within k_{n-1} of the ends of N_n-blocks, where it is defined arbitrarily,

(2) ϕ_n has the same distribution as (T, P) inside N_n-blocks,

(3) $\bar{f}_{N_n}((\psi_n(x)_1^{N_n}, \phi_n(x)_1^{N_n} < \dot{\varepsilon}_{n-1}$ for all sequences x having time-zero at the beginning of an N_n-block, except for a set of conditional measure ε_{n-1},

(4) the names within an N_n-block are determined by partitions of the factor space given by the product of $\bar{\bar{B}}$ and the independent processes generated by

$W_1 \times \cdots \times W_{n-1}$. Since $\Sigma \varepsilon_i < \infty$, there exists a limit transformation S', with S' Kakutani equivalent to S, as well as a limit partition Q'. Since (S', Q') is the $\bar{f}$-limit of the (S_n, Q_n), it must have the same distribution as (T, P).

5. Loosely Bernoulli and finitely fixed

Loosely Bernoulli (or finitely fixed) transformations are known to be exactly those which are Kakutani equivalent either to Bernoulli shifts or to circle rotations. The class of loosely Bernoulli transformations is the $\bar{f}$-closure of the Bernoulli shifts, and by Proposition 5, must be the same as the class of LBI processes. In fact one can see directly, as in [4], that loosely Bernoulli implies loosely block independent.

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